

# A LOWER BOUND FOR EIGENVALUES OF THE POLY-LAPLACIAN WITH ARBITRARY ORDER\*

QING-MING CHENG, XUERONG QI AND GUOXIN WEI

**ABSTRACT.** In this paper, we study eigenvalues of the poly-Laplacian with arbitrary order on a bounded domain in an  $n$ -dimensional Euclidean space and obtain a lower bound for eigenvalues, which gives an important improvement of results due to Levine and Protter [12]. In particular, the result of Melas [15] is included here.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$  be a bounded domain with piecewise smooth boundary  $\partial\Omega$  in an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Let  $\lambda_i$  be the  $i$ -th eigenvalue of the Dirichlet eigenvalue problem of the poly-Laplacian with arbitrary order:

$$\begin{cases} (-\Delta)^l u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Delta$  is the Laplacian in  $\mathbb{R}^n$  and  $\nu$  denotes the outward unit normal vector field of the boundary  $\partial\Omega$ . It is well known that the spectrum of this eigenvalue problem is real and discrete:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow +\infty,$$

where each  $\lambda_i$  has finite multiplicity which is repeated according to its multiplicity.

Let  $V(\Omega)$  denote the volume of  $\Omega$  and let  $B_n$  denote the volume of the unit ball in  $\mathbb{R}^n$ . When  $l = 1$ , the eigenvalue problem (1.1) is called a fixed membrane problem. In this case, one has the following Weyl's asymptotic formula

$$\lambda_k \sim \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty. \quad (1.2)$$

From the above asymptotic formula, one can obtain

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \rightarrow +\infty. \quad (1.3)$$

Pólya [18] proved that

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots, \quad (1.4)$$

---

2010 *Mathematics Subject Classification*: 35P15.

*Key words and phrases*: the eigenvalue problem, a lower bound for eigenvalues, the poly-Laplacian with arbitrary order.

\* Research partially supported by a Grant-in-Aid for Scientific Research from JSPS and NSFC.

if  $\Omega$  is a tiling domain in  $\mathbb{R}^n$ . Moreover, he proposed the following:

**Conjecture of Pólya.** *If  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ , then the  $k$ -th eigenvalue  $\lambda_k$  of the fixed membrane problem satisfies*

$$\lambda_k \geq \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots. \quad (1.5)$$

On the conjecture of Pólya, Berezin [4] and Lieb [14] gave a partial solution. In particular, Li and Yau [13] proved the following

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots. \quad (1.6)$$

The formula (1.3) shows that the result of Li and Yau is sharp in the sense of average. From this formula (1.6), one can derive

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad \text{for } k = 1, 2, \dots, \quad (1.7)$$

which gives a partial solution for the conjecture of Pólya with a factor  $\frac{n}{n+2}$ . Recently, Melas [15] have improved the estimate (1.6) to the following:

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{1}{24(n+2)} \frac{V(\Omega)}{I(\Omega)}, \quad \text{for } k = 1, 2, \dots, \quad (1.8)$$

where

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx$$

is called *the moment of inertia* of  $\Omega$ .

When  $l = 2$ , the eigenvalue problem (1.1) is called a clamped plate problem. For the eigenvalues of the clamped plate problem, it follows from Agmon [1] and Pleijel [17] that

$$\lambda_k \sim \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow +\infty. \quad (1.9)$$

This implies that

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \sim \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}, \quad k \rightarrow +\infty. \quad (1.10)$$

Furthermore, Levine and Protter [12] proved that the eigenvalues of the clamped plate problem satisfy

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}}. \quad (1.11)$$

The formula (1.10) shows that the coefficient of  $k^{\frac{4}{n}}$  is the best possible constant. Very recently, the first author and the third author [8] obtained the following estimate

which is an improvement of (1.11):

$$\begin{aligned} \frac{1}{k} \sum_{i=1}^k \lambda_i &\geq \frac{n}{n+4} \frac{16\pi^4}{(B_n V(\Omega))^{\frac{4}{n}}} k^{\frac{4}{n}} \\ &\quad + c_n \frac{n}{n+2} \frac{4\pi^2}{(B_n V(\Omega))^{\frac{2}{n}}} \frac{V(\Omega)}{I(\Omega)} k^{\frac{2}{n}} + d_n \left( \frac{V(\Omega)}{I(\Omega)} \right)^2, \end{aligned} \quad (1.12)$$

where  $c_n$  and  $d_n$  are constants depending only on the dimension  $n$ .

When  $l \geq 3$ , Levine and Protter [12] proved the following

$$\frac{1}{k} \sum_{i=1}^k \lambda_i \geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}}, \quad \text{for } k = 1, 2, \dots. \quad (1.13)$$

From the above formula, one can obtain

$$\lambda_k \geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}}, \quad \text{for } k = 1, 2, \dots. \quad (1.14)$$

In this paper we investigate eigenvalues of the Dirichlet eigenvalue problem (1.1) of Laplacian with any order. We give an important improvement of the result (1.13) due to Levine and Protter [12] by adding  $l$  terms of lower order of  $k^{\frac{2l}{n}}$  to its right hand side. In fact, we prove the following:

**Theorem 1.** *Let  $\Omega$  be a bounded domain in an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . Assume that  $\lambda_i$ ,  $i = 1, 2, \dots$ , is the  $i$ -th eigenvalue of the eigenvalue problem (1.1). Then the eigenvalues satisfy*

$$\begin{aligned} \frac{1}{k} \sum_{j=1}^k \lambda_j &\geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{2l}{n}} \\ &\quad + \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{(24)^{pn} \cdots (n+2p-2)} \frac{(2\pi)^{2(l-p)}}{(B_n V(\Omega))^{\frac{2(l-p)}{n}}} \left( \frac{V(\Omega)}{I(\Omega)} \right)^p k^{\frac{2(l-p)}{n}}. \end{aligned}$$

**Remark 1.** If we take  $l = 1$  in Theorem 1, then we obtain the inequality (1.8).

## 2. PROOF OF THEOREM

In this section, we firstly introduce some definitions and basic facts about the symmetric decreasing rearrangements. Next, we give the proof of Theorem 1.

For a bounded domain  $\Omega \subset \mathbb{R}^n$ , the *moment of inertia* of  $\Omega$  is defined by

$$I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x - a|^2 dx.$$

By translating the origin, we may assume that

$$I(\Omega) = \int_{\Omega} |x|^2 dx.$$

Let  $\Omega^*$  be the symmetric rearrangement of  $\Omega$ , that is,  $\Omega^*$  is the open ball centered at the origin with the same volume as  $\Omega$ . Then

$$\Omega^* = \left\{ x \in \mathbb{R}^n; |x| < \left( \frac{V(\Omega)}{B_n} \right)^{\frac{1}{n}} \right\}.$$

By using the symmetric rearrangement  $\Omega^*$  of  $\Omega$ , we have

$$I(\Omega) = \int_{\Omega} |x|^2 dx \geq \int_{\Omega^*} |x|^2 dx = \frac{n}{n+2} V(\Omega) \left( \frac{V(\Omega)}{B_n} \right)^{\frac{2}{n}}. \quad (2.1)$$

Let  $f$  be a nonnegative continuous function on  $\Omega$ . We consider its *distribution function*  $\mu_f(t)$  defined by

$$\mu_f(t) = \text{Vol}(\{x \in \Omega; f(x) > t\}).$$

The distribution function can be viewed as a function from  $[0, +\infty)$  to  $[0, V(\Omega)]$ . The *symmetric decreasing rearrangement*  $f^*$  of  $f$  is defined by

$$f^*(x) = \inf\{t \geq 0; \mu_f(t) < B_n |x|^n\}, \quad \text{for } x \in \Omega^*.$$

By definition, we know that  $f^*(x)$  is a radially symmetric function and

$$\text{Vol}(\{x \in \Omega; f(x) > t\}) = \text{Vol}(\{x \in \Omega^*; f^*(x) > t\}), \quad \forall t > 0.$$

Let  $f^*(x) = \phi(|x|)$ . Then one gets that  $\phi : [0, +\infty) \rightarrow [0, \sup f]$  is a decreasing function of  $|x|$ . We may assume that  $\phi$  is absolutely continuous. It is well known that

$$\int_{\Omega} f(x) dx = \int_{\Omega^*} f^*(x) dx = n B_n \int_0^{+\infty} s^{n-1} \phi(s) ds \quad (2.2)$$

and

$$\int_{\Omega} |x|^{2l} f(x) dx \geq \int_{\Omega^*} |x|^{2l} f^*(x) dx = n B_n \int_0^{+\infty} s^{n+2l-1} \phi(s) ds. \quad (2.3)$$

Good sources of further information on rearrangements are [3], [19].

One gets from the coarea formula that

$$\mu_f(t) = \int_t^{\sup f} \int_{\{f=s\}} |\nabla f|^{-1} d\sigma_s ds.$$

Since  $f^*$  is radial, we have

$$\begin{aligned} \mu_f(\phi(s)) &= \text{Vol}\{x \in \Omega; f(x) > \phi(s)\} = \text{Vol}\{x \in \Omega^*; f^*(x) > \phi(s)\} \\ &= \text{Vol}\{x \in \Omega^*; \phi(|x|) > \phi(s)\} = B_n s^n. \end{aligned}$$

It follows that

$$n B_n s^{n-1} = \mu'_f(\phi(s)) \phi'(s)$$

for almost every  $s$ . Putting  $\tau := \sup |\nabla f|$ , we obtain from the above equations and the isoperimetric inequality that

$$\begin{aligned} -\mu'_f(\phi(s)) &= \int_{\{f=\phi(s)\}} |\nabla f|^{-1} d\sigma_{\phi(s)} \geq \tau^{-1} \text{Vol}_{n-1}(\{f = \phi(s)\}) \\ &\geq \tau^{-1} n B_n s^{n-1}. \end{aligned}$$

Therefore, one obtains

$$-\tau \leq \phi'(s) \leq 0 \quad (2.4)$$

for almost every  $s$ .

In order to prove our theorem, we need the following lemma.

**Lemma 1.** *Let  $b \geq 1$ ,  $\eta, A > 0$  and  $\psi : [0, +\infty) \rightarrow [0, +\infty)$  be a decreasing, absolutely continuous function such that*

$$-\eta \leq \psi'(s) \leq 0, \quad A = \int_0^{+\infty} s^{b-1} \psi(s) ds.$$

For any positive integer  $l$ , let

$$A_l := \int_0^{+\infty} s^{b+2l-1} \psi(s) ds.$$

Then, we have

$$A_l \geq \frac{1}{b+2l} \left[ (bA)^{\frac{b+2l}{b}} \psi(0)^{-\frac{2l}{b}} + \sum_{p=1}^l \frac{(l+1-p)(bA)^{\frac{b+2(l-p)}{b}} \psi(0)^{\frac{2pb-2(l-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right].$$

*Proof.* Using the method of induction. Firstly, one can get from the lemma of [15] that

$$A_1 = \int_0^{+\infty} s^{b+1} \psi(s) ds \geq \frac{1}{b+2} \left[ (bA)^{\frac{b+2}{b}} \psi(0)^{-\frac{2}{b}} + \frac{A\psi(0)^2}{6\eta^2} \right]. \quad (2.5)$$

Secondly, we assume that

$$A_r \geq \frac{1}{b+2r} \left[ (bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right].$$

Since the formula (2.5) holds for any  $b \geq 1$ , we have

$$\begin{aligned} A_{r+1} &= \int_0^{+\infty} s^{b+2r+1} \psi(s) ds \\ &\geq \frac{1}{b+2r+2} \left\{ [(b+2r)A_r]^{\frac{b+2r+2}{b+2r}} \psi(0)^{-\frac{2}{b+2r}} + \frac{A_r \psi(0)^2}{6\eta^2} \right\} \\ &\geq \frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2} \left[ (bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} + \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2pb-2(r-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} \\ &\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r-p)}{b}} \psi(0)^{\frac{2(p+1)b-2(r-p)}{b}}}{6^{p+1} b \cdots (b+2p-2) \eta^{2p+2}} \\ &\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\psi(0)^{-\frac{2}{b+2r}}}{b+2r+2} \left[ (bA)^{\frac{b+2r}{b}} \psi(0)^{-\frac{2r}{b}} \right]^{\frac{b+2r+2}{b+2r}} \\
&\quad \times \left[ 1 + \sum_{p=1}^r \frac{(r+1-p)(bA)^{-\frac{2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} \\
&\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^{r+1} \frac{(r+2-p)(bA)^{\frac{b+2r-2p+2}{b}} \psi(0)^{\frac{2pb-2r+2p-2}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}} \\
&= \frac{(bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}}}{b+2r+2} \left[ 1 + \sum_{p=1}^r \frac{(r+1-p)(bA)^{-\frac{2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right]^{\frac{b+2r+2}{b+2r}} \\
&\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
&\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}}.
\end{aligned}$$

It follows from the Taylor formula that

$$\begin{aligned}
A_{r+1} &\geq \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
&\quad \times \left[ 1 + \frac{b+2r+2}{b+2r} \sum_{p=1}^r \frac{(r+1-p)(bA)^{-\frac{2p}{b}} \psi(0)^{\frac{2pb+2p}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \right] \\
&\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
&\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}} \\
&= \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
&\quad + \frac{1}{b+2r} \sum_{p=1}^r \frac{(r+1-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \\
&\quad + \frac{(bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}}{6(b+2r)(b+2r+2) \eta^2} + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
&\quad + \frac{1}{(b+2r)(b+2r+2)} \sum_{p=2}^r \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-4) \eta^{2p}} \\
&= \frac{1}{b+2r+2} (bA)^{\frac{b+2r+2}{b}} \psi(0)^{-\frac{2r+2}{b}} \\
&\quad + \left[ \frac{r}{b(b+2r)} + \frac{1}{(b+2r)(b+2r+2)} \right] \frac{1}{6\eta^2} (bA)^{\frac{b+2r}{b}} \psi(0)^{\frac{2b-2r}{b}}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{p=2}^r \left[ \frac{r+1-p}{b+2r} + \frac{(r+2-p)(b+2p-2)}{(b+2r)(b+2r+2)} \right] \frac{(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}} \\
& + \frac{(bA) \psi(0)^{2(r+1)}}{6^{r+1} b \cdots (b+2r+2) \eta^{2(r+1)}} \\
& \geq \frac{1}{b+2(r+1)} (bA)^{\frac{b+2(r+1)}{b}} \psi(0)^{-\frac{2(r+1)}{b}} \\
& + \frac{1}{b+2(r+1)} \sum_{p=1}^{r+1} \frac{(r+2-p)(bA)^{\frac{b+2(r+1-p)}{b}} \psi(0)^{\frac{2pb-2(r+1-p)}{b}}}{6^p b \cdots (b+2p-2) \eta^{2p}}.
\end{aligned}$$

This completes the proof of Lemma 1.  $\square$

*Proof of Theorem 1.* Let  $u_j$  be an orthonormal eigenfunction corresponding to the eigenvalue  $\lambda_j$ , that is,  $u_j$  satisfies

$$\begin{cases} (-\Delta)^l u_j = \lambda_j u_j, & \text{in } \Omega, \\ u_j = \frac{\partial u_j}{\partial \nu} = \cdots = \frac{\partial^{l-1} u_j}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \\ \int_{\Omega} u_i u_j = \delta_{ij}, & \text{for any } i, j. \end{cases} \quad (2.6)$$

Thus,  $\{u_j\}_{j=1}^{\infty}$  forms an orthonormal basis of  $L^2(\Omega)$ . We define a function  $\varphi_j$  by

$$\varphi_j(x) = \begin{cases} u_j(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \quad (2.7)$$

The Fourier transform  $\widehat{\varphi}_j(z)$  of  $\varphi_j(x)$  is then given by

$$\widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi_j(x) e^{i\langle x, z \rangle} dx = (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx. \quad (2.8)$$

We fix a  $k \geq 1$  and set

$$f(z) = \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2, \quad \text{for } z \in \mathbb{R}^n.$$

From Bessel's inequality, it follows that

$$\begin{aligned}
0 \leq f(z) &= \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2 = (2\pi)^{-n} \sum_{j=1}^k \left| \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 \\
&\leq (2\pi)^{-n} \int_{\Omega} |e^{i\langle x, z \rangle}|^2 dx = (2\pi)^{-n} V(\Omega).
\end{aligned} \quad (2.9)$$

By Parseval's identity, we have

$$\int_{\mathbb{R}^n} f(z) dz = \sum_{j=1}^k \int_{\mathbb{R}^n} |\widehat{\varphi}_j(z)|^2 dz = \sum_{j=1}^k \int_{\mathbb{R}^n} \varphi_j^2(x) dx = \sum_{j=1}^k \int_{\Omega} u_j^2(x) dx = k. \quad (2.10)$$

Furthermore, we deduce from integration by parts and Parseval's identity that

$$\begin{aligned}
\int_{\mathbb{R}^n} |z|^{2l} f(z) dz &= \sum_{j=1}^k \int_{\mathbb{R}^n} |z|^{2l} |\widehat{\varphi}_j(z)|^2 dz \\
&= \sum_{j=1}^k \int_{\mathbb{R}^n} |z|^{2l} \left| (2\pi)^{-n/2} \int_{\Omega} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} z_{r_1} \cdots z_{r_l} u_j(x) e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} u_j(x) \frac{\partial^l e^{i\langle x, z \rangle}}{\partial x_{r_1} \cdots \partial x_{r_l}} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| (2\pi)^{-n/2} \int_{\Omega} \frac{\partial^l u_j(x)}{\partial x_{r_1} \cdots \partial x_{r_l}} e^{i\langle x, z \rangle} dx \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left| \widehat{\frac{\partial^l u_j}{\partial x_{r_1} \cdots \partial x_{r_l}}} \right|^2 dz \\
&= \sum_{j=1}^k \sum_{r_1, \dots, r_l=1}^n \int_{\mathbb{R}^n} \left( \frac{\partial^l u_j}{\partial x_{r_1} \cdots \partial x_{r_l}} \right)^2 dx \\
&= \sum_{j=1}^k \int_{\Omega} u_j (-\Delta)^l u_j dx = \sum_{j=1}^k \lambda_j.
\end{aligned} \tag{2.11}$$

Since

$$\nabla \widehat{\varphi}_j(z) = (2\pi)^{-n/2} \int_{\Omega} i x u_j(x) e^{i\langle x, z \rangle} dx, \tag{2.12}$$

we obtain from Bessel's inequality

$$\sum_{j=1}^k |\nabla \widehat{\varphi}_j(z)|^2 \leq (2\pi)^{-n} \int_{\Omega} |i x e^{i\langle x, z \rangle}|^2 dx = (2\pi)^{-n} I(\Omega). \tag{2.13}$$

It follows from (2.9), (2.13) and the Cauchy-Schwarz inequality that

$$\begin{aligned}
|\nabla f(z)| &\leq 2 \left( \sum_{j=1}^k |\widehat{\varphi}_j(z)|^2 \right)^{1/2} \left( \sum_{j=1}^k |\nabla \widehat{\varphi}_j(z)|^2 \right)^{1/2} \\
&\leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)}
\end{aligned} \tag{2.14}$$

for every  $z \in \mathbb{R}^n$ .

Using the symmetric decreasing rearrangement  $f^*$  of  $f$  and noting that

$$f^*(x) = \phi(|x|), \quad \tau = \sup |\nabla f| \leq 2(2\pi)^{-n} \sqrt{V(\Omega) I(\Omega)} := \eta,$$

we obtain, from (2.4),

$$-\eta \leq -\tau \leq \phi'(s) \leq 0 \tag{2.16}$$



for almost every  $s$ . According to (2.2) and (2.10), we infer

$$k = \int_{\mathbb{R}^n} f(z) dz = \int_{\mathbb{R}^n} f^*(z) dz = nB_n \int_0^{+\infty} s^{n-1} \phi(s) ds. \quad (2.17)$$

From (2.3) and (2.11), we obtain

$$\sum_{j=1}^k \lambda_j = \int_{\mathbb{R}^n} |z|^{2l} f(z) dz \geq \int_{\mathbb{R}^n} |z|^{2l} f^*(z) dz = nB_n \int_0^{+\infty} s^{n+2l-1} \phi(s) ds. \quad (2.18)$$

Now, we can apply Lemma 1 to the function  $\phi$  with

$$b = n, \quad A = \frac{k}{nB_n}, \quad \eta = 2(2\pi)^{-n} \sqrt{V(\Omega)I(\Omega)}. \quad (2.19)$$

We conclude that

$$\begin{aligned} \sum_{j=1}^k \lambda_j &\geq \frac{nB_n}{n+2l} \left( \frac{k}{B_n} \right)^{\frac{n+2l}{n}} \phi(0)^{-\frac{2l}{n}} \\ &\quad + \frac{nB_n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2) \eta^{2p}} \left( \frac{k}{B_n} \right)^{\frac{n+2l-2p}{n}} \phi(0)^{\frac{2pn+2p-2l}{n}}. \end{aligned} \quad (2.20)$$

Note that  $0 < \phi(0) \leq \sup f \leq (2\pi)^{-n} V(\Omega)$ . Hence we consider the function  $F$  defined by

$$\begin{aligned} F(t) &= \frac{nB_n}{n+2l} \left( \frac{k}{B_n} \right)^{\frac{n+2l}{n}} t^{-\frac{2l}{n}} \\ &\quad + \frac{nB_n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2) \eta^{2p}} \left( \frac{k}{B_n} \right)^{\frac{n+2l-2p}{n}} t^{\frac{2pn+2p-2l}{n}}, \end{aligned} \quad (2.21)$$

for  $t \in (0, (2\pi)^{-n} V(\Omega)]$ . From (2.1), we have

$$\eta \geq (2\pi)^{-n} B_n^{-\frac{1}{n}} V(\Omega)^{\frac{n+1}{n}}. \quad (2.22)$$

By a direct calculation, one gets from  $B_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$  that

$$\frac{B_n^{\frac{4}{n}}}{(2\pi)^2} < \frac{1}{2}, \quad (2.23)$$

where  $\Gamma(\frac{n}{2})$  is the Gamma function. Thus, it follows from (2.22) and (2.23) that

$$\begin{aligned}
F'(t) &= \frac{2B_n t^{-\frac{n+2l}{n}}}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} \left[ -l + \sum_{p=1}^l \frac{(l+1-p)(pn+p-l)t^{\frac{2p(n+1)}{n}}}{6^p n \cdots (n+2p-2)\eta^{2p}} \left(\frac{k}{B_n}\right)^{-\frac{2p}{n}} \right] \\
&\leq \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[ -l + \sum_{p>\frac{l}{n+1}}^l \frac{(l+1-p)(pn+p-l)}{6^p n \cdots (n+2p-2)} \left(\frac{B_n^{\frac{4}{n}}}{(2\pi)^2}\right)^p \right] \\
&< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[ -l + \sum_{p>\frac{l}{n+1}}^l \frac{(l+1-p)(pn+p-l)}{(12)^p n \cdots (n+2p-2)} \right] \\
&< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[ -l + \frac{l(n+1-l)}{12n} + \sum_{p>\frac{l}{n+1}, p \neq 1}^l \frac{p^2 n(n+1)}{(12)^p n \cdots (n+2p-2)} \right] \\
&< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[ -l + \frac{l}{12} + \sum_{p>\frac{l}{n+1}, p \neq 1}^l \frac{p^2}{(12)^p} \right] \\
&< \frac{2B_n}{n+2l} \left(\frac{k}{B_n}\right)^{\frac{n+2l}{n}} t^{-\frac{n+2l}{n}} \left[ -l + \frac{l}{12} + \frac{1}{12} \right] < 0.
\end{aligned}$$

We obtain that  $F(t)$  is a decreasing function on  $(0, (2\pi)^{-n}V(\Omega)]$ . Then we can replace  $\phi(0)$  by  $(2\pi)^{-n}V(\Omega)$  in (2.20), namely

$$\begin{aligned}
\sum_{j=1}^k \lambda_j &\geq \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{n+2l}{n}} \\
&\quad + \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{6^p n \cdots (n+2p-2)\eta^{2p}} \frac{(V(\Omega))^{\frac{2pn+2p-2l}{n}}}{(2\pi)^{2pn+2p-2n} B_n^{\frac{2l-2p}{n}}} k^{\frac{n+2l-2p}{n}} \\
&= \frac{n}{n+2l} \frac{(2\pi)^{2l}}{(B_n V(\Omega))^{\frac{2l}{n}}} k^{\frac{n+2l}{n}} \\
&\quad + \frac{n}{n+2l} \sum_{p=1}^l \frac{(l+1-p)}{24^p n \cdots (n+2p-2)} \frac{(2\pi)^{2(l-p)}}{(B_n V(\Omega))^{\frac{2(l-p)}{n}}} \left(\frac{V(\Omega)}{I(\Omega)}\right)^p k^{\frac{n+2(l-p)}{n}}.
\end{aligned}$$

This completes the proof of Theorem 1.  $\square$

## REFERENCES

- [1] S. Agmon, *On kernels, eigenvalues and eigenfunctions of operators related to elliptic problems*, Comm. Pure Appl. Math. 18 (1965), 627-663.
- [2] M. S. Ashbaugh and R. D. Benguria, *Universal bounds for the low eigenvalues of Neumann Laplacian in  $N$  dimensions*, SIAM J. Math. Anal. 24 (1993), 557-570.

- [3] C. Bandle, *Isoperimetric inequalities and applications*, Pitman Monographs and Studies in Mathematics, vol. 7, Pitman, Boston, 1980.
- [4] F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat. 36 (1972), 1134-1167.
- [5] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.
- [6] Q. -M. Cheng, G. Huang and G. Wei, *Estimates for lower order eigenvalues of a clamped plate problem*, Calc. Var. Partial Differential Equations 38 (2010), 409-416.
- [7] Q. -M. Cheng, T. Ichikawa and S. Mametsuka, *Inequalities for eigenvalues of Laplacian with any order*, Commun. Contemp. Math. 11 (2009), 639-655.
- [8] Q. -M. Cheng and G. Wei, *A lower bound for eigenvalues of a clamped plate problem*, <http://arxiv.org/abs/0908.3829>.
- [9] Q. -M. Cheng and H. C. Yang, *Inequalities for eigenvalues of a clamped plate problem*, Trans. Amer. Math. Soc. 358 (2006), 2625-2635.
- [10] Q. -M. Cheng and H. C. Yang, *Estimates for eigenvalues on Riemannian manifolds*, J. Differential Equations 247 (2009), 2270-2281.
- [11] A. Laptev, *Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces*, J. Funct. Anal. 151 (1997), 531-545.
- [12] H. A. Levine and M. H. Protter, *Unrestricted lower bounds for eigenvalues for classes of elliptic equations and systems of equations with applications to problems in elasticity*, Math. Methods Appl. Sci. 7 (1985), no. 2, 210-222.
- [13] P. Li and S. T. Yau, *On the Schrödinger equations and the eigenvalue problem*, Comm. Math. Phys. 88 (1983), 309-318.
- [14] E. Lieb, *The number of bound states of one-body Schrödinger operators and the Weyl problem*, Proc. Sym. Pure Math. 36 (1980), 241-252.
- [15] A. D. Melas, *A lower bound for sums of eigenvalues of the Laplacian*, Proc. Amer. Math. Soc. 131 (2003), 631-636.
- [16] L. E. Payne, G. Pólya and H. F. Weinberger, *On the ratio of consecutive eigenvalues*, J. Math. and Phys. 35 (1956), 289-298.
- [17] A. Pleijel, *On the eigenvalues and eigenfunctions of elastic plates*, Comm. Pure Appl. Math., 3 (1950), 1-10.
- [18] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc., 11 (1961), 419-433.
- [19] G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics*, Annals of mathematics studies, number 27, Princeton university press, Princeton, New Jersey, 1951.
- [20] Q. L. Wang, C. Y. Xia, *Universal bounds for eigenvalues of the biharmonic operator on Riemannian manifolds*, J. Funct. Anal. 245 (2007), 334-352.

QING-MING CHENG, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, SAGA UNIVERSITY, SAGA 840-8502, JAPAN, CHENG@MS.SAGA-U.AC.JP

XUERONG QI, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ENGINEERING, SAGA UNIVERSITY, SAGA 840-8502, JAPAN, QIXUERONG609@GMAIL.COM

GUOXIN WEI, SCHOOL OF MATHEMATICAL SCIENCES, SOUTH CHINA NORMAL UNIVERSITY, 510631, GUANGZHOU, CHINA, WEIGX03@MAILS.TSINGHUA.EDU.CN